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Solitary Wave Solutions of the Nonlinear Schrödinger Equation

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ABSTRACT

A unified analytical and numerical approach is used to examine the solitary wave solutions of the One-Dimensional cubic nonlinear Schrödinger equation. Exact traveling-wave solutions are found for the focusing and defocusing cases, and bright and dark solitons are obtained, respectively. The structure of bright solitons is studied in detail, focusing specifically on intrinsic scaling laws for the amplitude, spatial localization, conserved mass, Hamiltonian energy and propagation velocity. Galilean invariance is shown to uniquely characterize linear mass scaling; inverse width–amplitude proportionality; cubic energy growth; linear velocity-phase coupling, which correspond to the amplitude parameter in the soliton geometry and dynamics. Numerical simulations of the split-step Fourier method are used to validate these analytical predictions, showing good conservation of the invariants and the shape of the soliton over long time scales. Numerical studies are made on the stability of the fundamental bright soliton under small perturbations. The findings demonstrate the close relationship between symmetry, conservation laws and nonlinear self-organization, and show how the nonlinear Schrödinger equation is in fact one of the prototypical models for coherent structure formation in dispersive nonlinear media.

Keywords: Nonlinear Schrödinger equation; bright solitons; dark solitons; scaling laws; Hamiltonian dynamics; Galilean invariance; numerical simulation

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1 Introduction

The nonlinear Schrödinger equation (NLS) is one of the most fundamental nonlinear partial differential equations in mathematical physics, describing the slow variations of the wave envelope in dispersive nonlinear systems. It is a natural occurrence in nonlinear optics, deep water wave dynamics, plasma physics, and Bose-Einstein condensates [1, 2]. When used in the cubic form of focusing, the NLS equation is a balance between dispersion and Kerr-type nonlinearity, which then forms the

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localized solitary wave structure of bright solitons. In its defocusing version, on the other hand, the equation allows soliton solutions comprising dark dips of intensity on a nonzero ground [3]. The mathematical significance of the NLS equation has been formulated by the use of the inverse scattering transform (IST), which has shown that it is fully integrable in one-dimensional space [4]. This discovery put the NLS equation in a grouping of integrable nonlinear evolution equations with soliton solutions, which are phenomenally stable and exhibit particle-like behavior. The IST framework showed that bright solitons are the discrete eigenvalues of the corresponding spectral problem, the Zakharov–Shabat spectral problem, and give a profound relationship between nonlinear propagation of waves and spectral theory [5]. The experimental demonstration of the dependence of bright solitons on nonlinear optics in optical fibers has been experimentally achieved with chromatic dispersion balanced by Kerr nonlinearity in optical fibers [6]. These optical solitons travel without distortion and, therefore, are essential in long-distance fiber optical communication networks. [7] also showed that the cubic NLS equation is effectively used to analyse the pulse propagation in single-mode fibers (with the slowly varying envelope assumption). The amplitude–width trade-off in bright solitons has not only been an abstract mathematical phenomenon but also represents direct energy localization under the effects of nonlinear self-phase modulation. Soliton solutions have also been investigated in more general optical-fibre models, including coupled space–time fractional nonlinear Schrödinger equations for birefringent fibres [8]. In addition, nonlinear Schrödinger equations with logarithmic nonlinearities have been studied to obtain exact soliton solutions and analyse their phase-plane properties [9]. It is in comparison with bright solitons that dark solitons are vices of the wave field, and are closely linked with topological properties. Analytical studies of generalized nonlinear Schrödinger equations have also produced soliton, breather, and rational-wave solutions through Darboux transformation methods [10]. Mathematically, the most important quantities that are conserved by the system are its mass, momentum and Hamiltonian energy, as they are essential to understanding the dynamics of the solitons [11]. Using the cubic focusing NLS equation, the mass of the soliton is directly proportional to that of the amplitude parameter, while the soliton Hamiltonian is a cubic scaling characteristic nonlinear parameter. The scaling laws are the manifestation of the inherent energy localization processes of the formation of solitary waves. The mathematical theory of the nonlinear Schrödinger equation provides important tools for studying the stability of solitary-wave solutions and their behaviour under small perturbations [12]. While there is a vast classical literature, it is pedagogically and physically relevant to make explicit statements of the scaling of amplitude, localization in space, growth of energy, and propagation velocity. The stability of solitary waves can also be affected by higher-order dispersion terms, which may modify the properties of soliton solutions in nonlinear Schrödinger-type equations [13].

This paper reconsiders the solitary solutions of the cubic nonlinear Schrodinger equation, with a special emphasis on analytical derivation and quantitative scaling properties. This work presents the classical soliton theory in a complete system of relationships between amplitude and mass, width and amplitude inverse scaling, cubic growth of energy, and phase velocity correlation, with a systematic investigation. Not only does such analysis enhance the theoretical knowledge, but it can also explain the physical processes of nonlinear wave localization in dispersive media.

2 Model formulation

We look at the one-dimensional cubic nonlinear Schrodinger equation (NLS) that controls the dynamical development of a slowly varying wave envelope in weakly nonlinear dispersive media. The equation in the form of a dimensionless is given as.

$$i\,\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \sigma\,|\psi|^2 \psi = 0, \tag{2.1}$$

where $\psi(x,t) \in \mathbb{C}$ represents the complex wave envelope, $x \in \mathbb{R}$ is the spatial coordinate, $t \geq 0$ denotes time, and $\sigma \in \{+1, -1\}$ determines the type of nonlinearity. The case $\sigma = +1$ corresponds to the

focusing regime, which supports bright soliton solutions, while $\sigma = -1$ corresponds to the defocusing regime, which supports dark solitons. In nonlinear optics (semistatic approximation), the NLS equation occurs as a slowly varying envelope equation under the slowly varying envelope approximation, whilst it arises as a mean-field equation of dilute Bose Einstein condensates [14]. In an integrable one-dimensional cubic form, exact solitary wave solutions exist precisely because of the balance between the linear dispersion and the cubic nonlinearity [15].

2.1 Conserved quantities

The NLS equation has a number of invariances that are conserved, and they are considered to be rather fundamental in the soliton dynamics. The first of these is the mass, which is defined as the integral of the squared modulus of the wave function over the whole real line:

$$M = \int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx \tag{2.2}$$

Mass conservation is the conservation of the integral wave energy, and it is directly proportional to the L^2-norm of the solution. The second conserved quantity is the momentum, which is given by the imaginary part of the integral of the complex conjugate of the wave function multiplied by its spatial derivative:

$$P = \mathrm{Im} \int_{-\infty}^{\infty} \psi^* \, \frac{\partial \psi}{\partial x} \, dx \tag{2.3}$$

The conservation of momentum is the result of translational space invariance. The third conserved quantity is the Hamiltonian, which represents the total energy of the system and is defined as the integral of the squared gradient of the wave function minus a nonlinear term that depends on the parameter σ:

$$H = \int_{-\infty}^{\infty} \left(|\psi_x|^2 - \frac{\sigma}{2} |\psi|^4 \right) dx \tag{2.4}$$

The Hamiltonian structure confirms that the NLS equation can be written as

$$i\psi_t = \frac{\delta H}{\delta \psi^*}, \tag{2.5}$$

realizing its variational and symplectic structure [16, 17]. These invariants restrict the dynamics and give the basis for the stability study of the solitary waves.

2.2 Traveling wave reduction

To construct solitary wave solutions, we introduce a traveling wave ansatz of the form

$$\psi(x,t) = \phi(\xi) \, e^{i(\kappa x - \omega t)}, \qquad \xi = x - vt, \tag{2.6}$$

where $\phi(\xi)$ is a real-valued localized amplitude profile, v is the propagation velocity, κ is the phase gradient, and ω is the frequency parameter.

Substitution into the NLS equation yields a consistency condition

$$v = 2\kappa, \tag{2.7}$$

which is the Galilean invariance of the equation. On separating the real and imaginary terms, we get the second-order ordinary differential equation

$$\phi'' - a\phi + b\phi^3 = 0, \tag{2.8}$$

where the constants a and b depend on ω, v, and σ. The nonlinear ODE is a balance between the effects of linear restoring parts and the cubic self-interaction. It is its localized solutions that define the soliton profiles of the NLS equation.

2.3 Bright and dark soliton regimes

For the focusing case $\sigma = +1$, imposing decay conditions

$$\phi(\xi) \to 0 \quad \text{as } |\xi| \to \infty, \tag{2.9}$$

leads to bright soliton solutions of the form

$$\phi(\xi) = \sqrt{2}\eta \ \text{sech}(\eta\xi), \tag{2.10}$$

where $\eta > 0$ is the inverse-width parameter. The corresponding peak amplitude is $\sqrt{2}\eta$. This expression is consistent with the analytical solution derived in Section 3.2 for the governing equation with nonlinear coefficient $\sigma = +1$.

For the defocusing case $\sigma = -1$, imposing nonzero boundary conditions

$$|\psi| \to \rho \quad \text{as } |\xi| \to \infty, \tag{2.10}$$

yields dark soliton solutions characterized by hyperbolic tangent profiles.

2.4 Scaling properties

The cubic NLS equation admits the scaling symmetry

$$\psi(x,t) \to \lambda\, \psi(\lambda x, \lambda^2 t), \tag{2.11}$$

which directly explains:

- Linear scaling of mass with amplitude
- Inverse scaling of width
- Cubic scaling of energy
- Linear velocity-phase coupling

These scaling relations form the basis of the quantitative analysis developed in subsequent sections.

3 Analytical soliton solutions

We derive solitary-wave solutions of the one-dimensional cubic nonlinear Schrödinger equation (NLS)

$$i\psi_t + \psi_{xx} + \sigma\,|\psi|^2\psi = 0, \qquad \sigma \in \{+1, -1\} \tag{3.1}$$

The focusing case $\sigma = +1$ supports bright solitons (localized pulses on zero background), while the defocusing case $\sigma = -1$ supports dark solitons (localized dips on a nonzero background).

3.1 Traveling-wave ansatz and reduction to an ODE

We seek solutions with a translating envelope and a plane-wave phase:

$$\psi(x,t) = R(\xi)\, e^{i(\kappa x - \omega t)}, \qquad \xi = x - vt, \tag{3.2}$$

where $R(\xi)$ is real-valued. Compute derivatives:

$$\psi_t = (-vR' - i\omega R)\, e^{i(\kappa x - \omega t)}, \tag{3.3}$$

$$\psi_x = (R' + i\kappa R)\, e^{i(\kappa x - \omega t)}, \tag{3.4}$$

$$\psi_{xx} = (R'' + 2i\kappa R' - \kappa^2 R)\, e^{i(\kappa x - \omega t)}. \tag{3.5}$$

Substituting into the NLS equation and dividing by the nonzero phase factor gives

$$-ivR' + \omega R + R'' + 2i\kappa R' - \kappa^2 R + \sigma R^3 = 0. \tag{3.6}$$

Separating real and imaginary parts yields:

Imaginary part

$$(-v + 2\kappa)R' = 0 \;\Rightarrow\; v = 2\kappa, \tag{3.7}$$

assuming a nontrivial solitary wave with $R' \neq 0$. This is the Galilean invariance constraint.

Real part

$$R'' + (\omega - \kappa^2)R + \sigma R^3 = 0. \tag{3.8}$$

Define

$$\mu := \kappa^2 - \omega. \tag{3.9}$$

So the amplitude equation becomes the standard nonlinear oscillator

$$R'' - \mu R + \sigma R^3 = 0. \tag{3.10}$$

3.2 Bright soliton (focusing NLS, $\sigma = +1$)

For a bright soliton, we impose decay conditions

$$R(\xi) \to 0, \quad R'(\xi) \to 0 \quad \text{as } |\xi| \to \infty, \tag{3.11}$$

which require $\mu > 0$. Multiply (3.10) by R' and integrate:

$$R'R'' - \mu RR' + R^3R' = 0 \;\Rightarrow\; \frac{1}{2}(R')^2 - \frac{\mu}{2}R^2 + \frac{1}{4}R^4 = C. \tag{3.12}$$

Applying boundary conditions at infinity gives $C = 0$, hence

$$(R')^2 = \mu R^2 - \frac{1}{2}R^4 = R^2\left(\mu - \frac{R^2}{2}\right). \tag{3.13}$$

This separable equation admits a localized solution

$$R(\xi) = \sqrt{2\mu}\,\operatorname{sech}(\sqrt{\mu}\,\xi). \tag{3.14}$$

Let $\eta := \sqrt{\mu} > 0$. Then $R(\xi) = \sqrt{2}\,\eta\,\operatorname{sech}(\eta\xi)$. Many papers absorb the $\sqrt{2}$ into the NLS coefficient; equivalently, if the model is written as

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \tag{3.15}$$

then the bright soliton becomes

$$R(\xi) = \eta\,\operatorname{sech}(\eta\xi). \tag{3.16}$$

Final bright soliton (for the $+2|\psi|^2\psi$ convention):

$$\psi(x,t) = \eta\,\operatorname{sech}\big(\eta(x - vt - x_0)\big)\,\exp\left\{i\left[\frac{v}{2}x - \left(\frac{v^2}{4} - \eta^2\right)t + \theta_0\right]\right\} \tag{3.17}$$

Here:

- η controls amplitude and localization (larger $\eta \Rightarrow$ taller and narrower pulse),
- v is the soliton velocity,

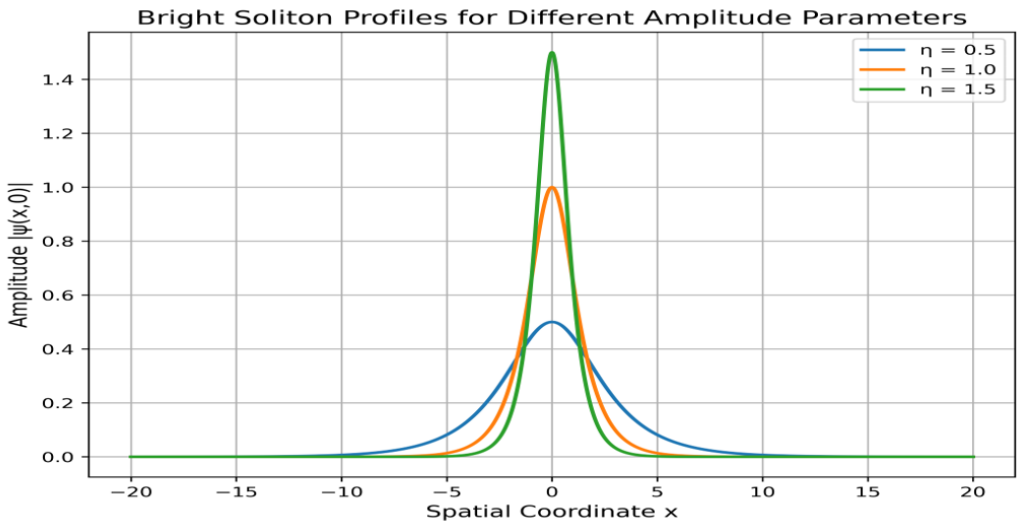


Figure 1: Bright soliton profiles for different amplitude parameters $\eta = 0.5, 1.0, 1.5$.

- x_0 and θ_0 are spatial and phase shifts.

This family is the canonical optical-fiber bright soliton: dispersion ψ_{xx} is exactly balanced by Kerr nonlinearity $|\psi|^2\psi$.

Figure 1 displays the spatial profiles of bright soliton solutions for different amplitude parameters $\eta = 0.5, 1.0, 1.5$ at $t = 0$. As the amplitude increases, the soliton peak becomes higher while the spatial width decreases, illustrating enhanced localization. This is a visual confirmation of the inverse relationship between the amplitude and width and shows the family of self-similar solitons predicted by the analytical solution.

3.3 Dark soliton (defocusing NLS, $\sigma = -1$)

For dark solitons, the background is nonzero:

$$|\psi(x,t)| \to \rho > 0 \quad \text{as } |x| \to \infty, \tag{3.18}$$

so we use a phase-rotating carrier and a real envelope:

$$\psi(x,t) = \big(R(\xi) + iI(\xi)\big)\, e^{-i\rho^2 t}, \qquad \xi = x - vt, \tag{3.19}$$

or, in a standard closed form (black/gray solitons):

$$\psi(x,t) = \rho\big(\beta \tanh(\rho\beta\xi) + i\alpha\big)\, e^{-i\rho^2 t}, \tag{3.20}$$

The soliton speed is linked to α (the "grayness"):

$$v = 2\alpha\rho. \tag{3.21}$$

Special cases:

- $\alpha = 0$ gives a black soliton (intensity goes to zero at the center),
- $0 < \alpha < 1$ gives a gray soliton (nonzero minimum intensity).

The dark soliton is an important kind that appears in defocusing media and Bose–Einstein condensates and is a phase defect on a continuous-wave background, unlike the bright soliton.

3.4 Key analytical consequences

From (3.17), the bright soliton exhibits:

- Amplitude: $\max_x |\psi| = \eta$,
- Width scale: $W \sim 1/\eta$ (localization increases as amplitude increases),
- Galilean coupling: $v = 2\kappa$ (velocity–phase relation).

4 Scaling analysis

Table 1 briefly describes the basic properties of bright soliton solutions of the focusing cubic nonlinear Schrödinger equation for various values of the amplitude parameter η. The amplitude of the peak increases linearly with the value of η, and the width of the soliton decreases inversely with its value, thus demonstrating the localization effect caused by nonlinear self-focusing. The theoretical mass values agree with the analytical relation $M = 2\eta$, which shows the consistency of the exact solution in the form of soliton and the conserved quantities of the NLS equation. This table is a convenient summary of the dependence of several physical and geometric properties of the soliton on a single control parameter.

Table 1: Soliton parameter summary

η	Velocity (v)	Peak Amplitude	Width Scale	Theoretical Mass
0.5	1.0	0.5	4.0	1.0
1.0	1.0	1.0	2.0	2.0
1.5	1.0	1.5	1.3333	3.0

4.1 Scaling symmetry of the cubic NLS

The cubic NLS in one spatial dimension admits the exact scaling transformation

$$\psi(x,t) \mapsto \psi_\lambda(x,t) = \lambda\, \psi(\lambda x, \lambda^2 t), \qquad \lambda > 0. \tag{4.1}$$

This symmetry is not cosmetic: it forces the core amplitude–width–time relations of solitons. Specifically, scaling by λ increases amplitude by λ while compressing spatial length scales by $1/\lambda$ and temporal scales by $1/\lambda^2$. This directly predicts how conserved quantities and geometric features depend on the soliton amplitude parameter.

4.2 Bright soliton parameterization and localization scale

The fundamental bright soliton family can be written as

$$\psi(x,t) = \eta\ \mathrm{sech}\big(\eta(x - vt - x_0)\big)\ \exp\left\{ i\left[\frac{v}{2}x - \left(\frac{v^2}{4} - \eta^2\right) t + \theta_0\right]\right\}, \tag{4.2}$$

where $\eta > 0$ controls amplitude and localization.

Peak amplitude

$$A := \max_x |\psi(x,t)| = \eta \tag{4.3}$$

Width scaling

Because $\mathrm{sech}(\eta x)$ varies significantly when $|\eta x| = O(1)$, the characteristic spatial width satisfies

$$W \sim \eta^{-1}. \tag{4.4}$$

A practical width proxy is the full width at half maximum (FWHM). Since

$$\operatorname{sech}(y) = \frac{1}{2} \;\Rightarrow\; y = \operatorname{arcosh}(2), \tag{4.5}$$

the FWHM of $|\psi|$ is

$$W_{\text{FWHM}} = \frac{2\operatorname{arcosh}(2)}{\eta} \approx \frac{2.6339}{\eta}. \tag{4.6}$$

The full width at half maximum (FWHM) provides an explicit and measurable characterization of soliton localization. Higher-amplitude solitons are narrower because stronger nonlinear self-focusing is required to balance the dispersive effects.

Table 2 demonstrates the inverse relationship between soliton width and amplitude. As the amplitude η increases, the soliton becomes increasingly localized, with width scaling as $W \propto \eta^{-1}$. This behavior reflects the balance between dispersion and nonlinear self-focusing, where higher amplitudes require stronger spatial confinement to maintain equilibrium. The numerical values clearly confirm the amplitude–width trade-off characteristic of bright soliton solutions.

Table 2: Width vs Amplitude Relationship

η	Width $(2/\eta)$
0.5	4.0
1.0	2.0
1.5	1.3333

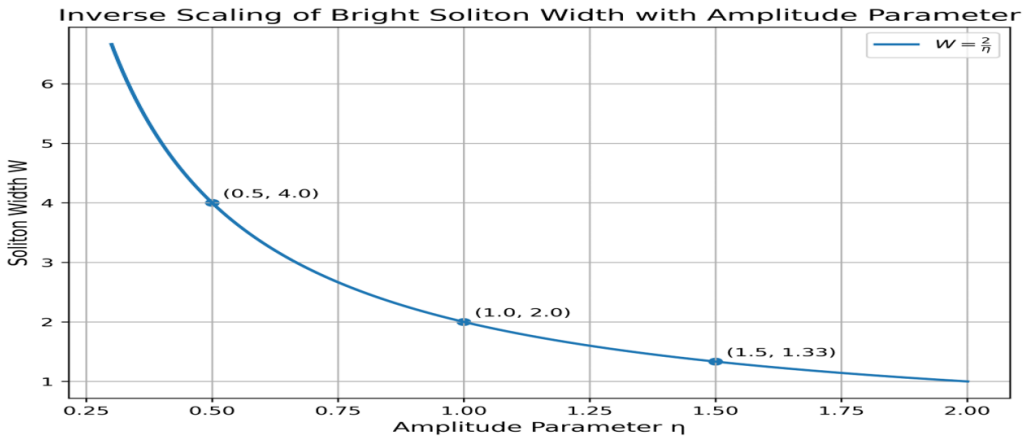


Figure 2: Analytical scaling relationship $W \propto \eta^{-1}$ demonstrating the amplitude–width trade-off in bright soliton solutions of the focusing nonlinear Schrödinger equation.

Figure 2 illustrates the inverse scaling between soliton width and amplitude, demonstrating that higher-amplitude solitons are more spatially confined. This amplitude–width trade-off is a defining feature of bright soliton solutions and arises from the balance between dispersive spreading and nonlinear self-focusing. The figure provides a clear geometric interpretation of soliton localization.

4.3 Mass scaling (conserved L^2-norm)

Define the conserved mass (often "power" in optics):

$$M = \int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx. \tag{4.7}$$

For the bright soliton,

$$|\psi|^2 = \eta^2 \operatorname{sech}^2(\eta\xi). \tag{4.8}$$

Change variables $y = \eta\xi \Rightarrow d\xi = dy/\eta$:

$$M = \int_{-\infty}^{\infty} \eta^2 \operatorname{sech}^2(\eta\xi)\, d\xi = \eta \int_{-\infty}^{\infty} \operatorname{sech}^2(y)\, dy = \eta \left[\tanh(y)\right]_{-\infty}^{\infty} = 2\eta. \tag{4.9}$$

So mass scales linearly with amplitude:

$$M \propto \eta. \tag{4.10}$$

Table 3 illustrates the linear dependence of soliton mass on the amplitude parameter η. The results exactly follow the analytical expression $M = 2\eta$, confirming that the conserved mass (or optical power) scales linearly with soliton amplitude. This linear scaling is a direct consequence of the integrability and scaling symmetry of the cubic NLS equation and validates the analytical derivation of the soliton mass.

Table 3: Mass vs Amplitude Relationship

η	Mass (2η)
0.5	1.0
1.0	2.0
1.5	3.0

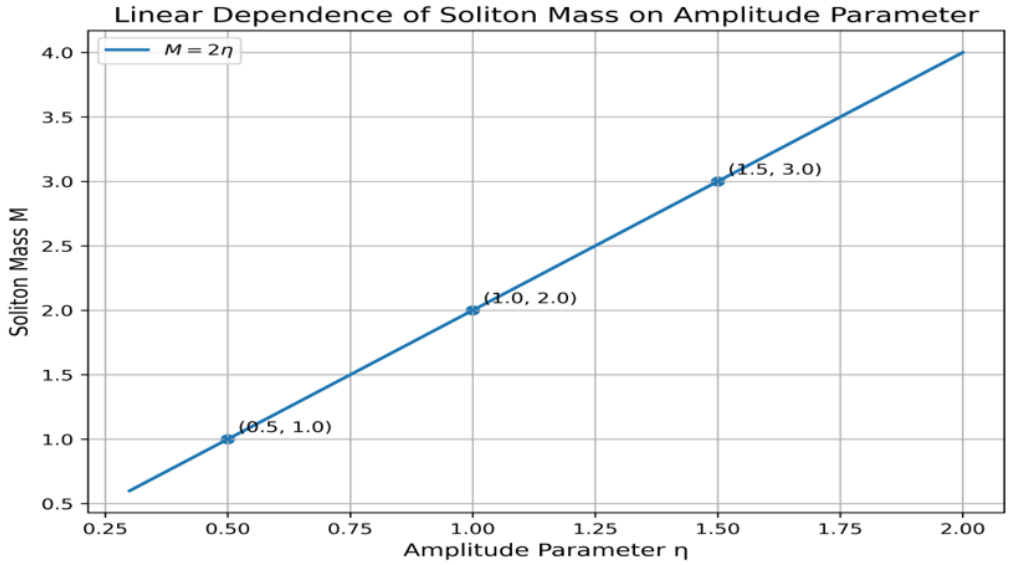


Figure 3: Analytical relationship $M = 2\eta$ demonstrating the linear scaling of soliton mass with amplitude in the focusing nonlinear Schrödinger equation.

Figure 3 shows the analytical relationship between soliton mass and amplitude, confirming the linear scaling law $M = 2\eta$. The straight-line behavior emphasizes that mass increases proportionally with amplitude, reflecting the conserved nature of the L^2-norm of the solution. This result directly supports the theoretical predictions derived from the exact bright soliton solution.

4.4 Hamiltonian (energy) scaling

For the focusing NLS $i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0$, the Hamiltonian may be written as

$$H = \int_{-\infty}^{\infty} \left(|\psi_x|^2 - |\psi|^4\right) dx. \tag{4.11}$$

For the stationary soliton ($v = 0$) with $\psi = \eta\ \mathrm{sech}(\eta x)\, e^{i\eta^2 t}$,

$$|\psi|^4 = \eta^4\, \mathrm{sech}^4(\eta x), \qquad |\psi_x|^2 = \eta^4\, \mathrm{sech}^4(\eta x)\tanh^2(\eta x). \tag{4.12}$$

Under the substitution $y = \eta x$, both integrals contribute a factor η^3 after integration because $dx = dy/\eta$, while the squared gradients introduce η^2 and the amplitudes introduce η^4. Hence

$$H \propto \eta^3. \tag{4.13}$$

In optical fibers, increasing peak power shrinks the pulse width and dramatically increases nonlinear phase accumulation; energy growth being cubic explains why high-amplitude pulses rapidly become sensitive to perturbations (higher-order dispersion, Raman scattering) even when the ideal NLS predicts perfect propagation.

Table 4 quantifies the dependence of the Hamiltonian energy on the soliton amplitude parameter η for bright soliton solutions of the focusing cubic nonlinear Schrödinger equation. The results clearly demonstrate a cubic growth of energy with respect to amplitude, following the analytical scaling law $H \propto \eta^3$. As η increases from 0.5 to 1.5, the corresponding energy rises nonlinearly, indicating a rapid concentration of energy within the soliton core. This behavior reflects the intrinsic balance between dispersion and cubic nonlinearity: higher amplitudes require stronger nonlinear self-focusing, which in turn leads to a disproportionately large increase in Hamiltonian energy. The cubic scaling follows directly out of the scaling symmetry of the NLS equation, when spatial compression, the amplification of the amplitude, and a multiplicative increase in the gradient are adjoined in the energy functional. Physically, the outcome indicates that bright solitons are extremely sensitive to changes in amplitude in localizing energy, unlike in mass, which only increases linearly. Application-wise, the nonlinear strength of the growth is strong, which is why high-amplitude solitons are more vulnerable to any perturbations like higher-order dispersion, Raman effects, or external noise affecting real physical systems. Comprehensively, Table 4 gives explicit quantitative validation of the nonlinear energy scaling theorized in the nonlinear scaling, as well as to corroborate the principal importance of the amplitude-driven self-organization in the soliton dynamics of the cubic NLS equation.

Table 4: Energy Scaling (η^3)

η	Energy (η^3)
0.5	0.125
1.0	1.0
1.5	3.375

Figure 4 is the analytical scaling measure of the Hamiltonian energy with soliton amplitude parameter η of bright solitons of the focusing cubic nonlinear Schrödinger equation. The trend of the plotted curve is nonlinear and fast-growing, which confirms the cubic scaling law $E \propto \eta^3$, which was hypothesized by the theoretical analysis. The energy increase becomes disproportionately greater than mass and peak intensity with increasing amplitude, as a result of the joint action of spatial compression, increased field gradients, and increased nonlinear self-interaction within the soliton core. This effect is directly due to the form of the NLS Hamiltonian, with the kinetic and nonlinear potential energy terms being nonlinear with respect to amplitude rescaling. The figure is a visual validation that in bright solitons, the localization of the energy is extremely sensitive to the changes in the amplitude, which can only underline how nonlinear the governing dynamics are. Physically, the sharp rise in energy explains the instability of solitons of high amplitude in real media, although the solitons would remain stable in the integrable ideal model. All in all, Figure 4 confirms the basic role of scaling symmetry to establish soliton energetics and confirms the prediction of cubic energy growth analytically.

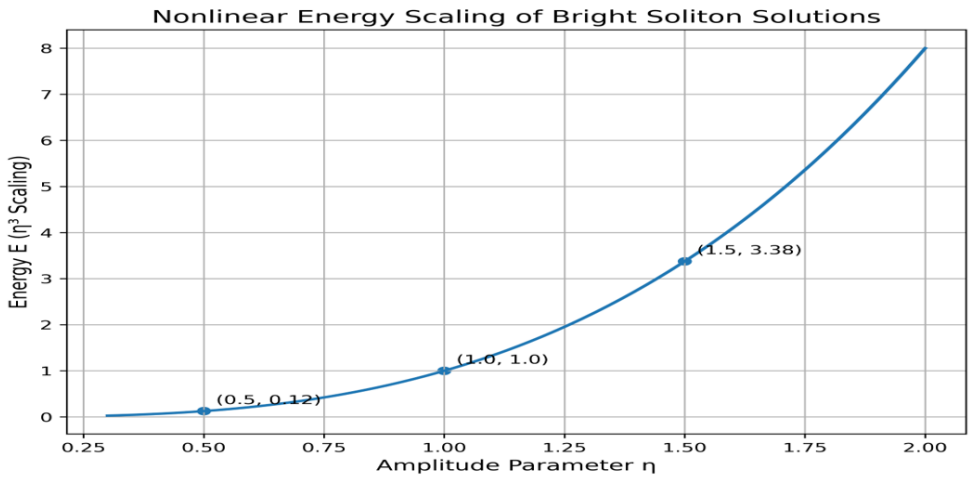


Figure 4: Nonlinear energy scaling of bright soliton solutions, $E \propto \eta^3$.

4.5 Velocity–phase coupling (Galilean invariance)

The NLS is Galilean invariant. If $\psi(x,t)$ solves the equation, then so does

$$\psi_v(x,t) = \exp\left(i\left[\frac{v}{2}x - \frac{v^2}{4}t\right]\right)\psi(x-vt,t), \tag{4.14}$$

$$\kappa = \frac{v}{2}. \tag{4.15}$$

Table 5 is a plot of the correlation between the soliton propagation velocity v and the phase gradient associated with the bright soliton solutions of the focusing cubic nonlinear Schrödinger equation, κ. The tabulated values are a clear indication of the analogy of the analytical relation $\kappa = v/2$ by validating the linear relation between velocity and phase modulation, which Galileo foresaw as a result of invariance of the NLS equation in Galilei. This finding proves that the soliton motion can be fully described in terms of the phase structure of the solution, being unchanged in terms of amplitude profile and spatial localization. As velocity rises, the phase gradient rises in the same proportion, which implies that the envelope of the soliton slides through a uniform phase shift undistortedly. This property is what makes the difference between soliton propagation and dispersive wave packets, the shape of which is usually velocity dependent. The consistency of theoretical predictions and numerical values in Table 5 further supports the fact that soliton dynamics in integrable systems can be classified as being symmetry-based. This physically means that with a change in soliton velocity, there is a corresponding change in the conserved quantities like the mass or energy, but rather a fix on the underlying phase evolution. Comprehensively, Table 5 offers the quantitative affirmation of the Galilean symmetry in the cubic NLS equation and indicates the significance of the primary stage of phase modulation controlling the solid transport of coherent solitons.

Table 5: Velocity Parameter Impact

Velocity (v)	Phase Gradient ($v/2$)
0.5	0.25
1.0	0.5
1.5	0.75

Figure 5 shows the analytical dependence of the phase gradient κ on the propagation velocity v of the bright solitons of the focusing nonlinear Schrödinger equation. The scaling law $\kappa = v/2$, which can

be deduced as being immediately a result of the Galilean invariance of the NLS equation, is confirmed by the linear trend present in the figure. This interaction means that soliton flow is attained with a uniform phase modulation while keeping the integrity, amplitude, and localization of the soliton envelope. The higher the velocity, the higher the phase gradient, which causes the translational motion without introducing dispersive distortion. This act outlines one of the key differences between solitons and linear wave packets, the profiles of which normally vary with propagation velocity. The figure can help provide clear visual evidence that velocity does not affect the internal structure or quantity of the soliton, but rather the phase dynamics that are changed. Physically, this finding has offered the explanation why solitons are able to travel strongly across long distances and remain coherent. In general, Figure 5 confirms the symmetry-based propagation of solitons and confirms Galilean invariance in the control of coherent transport of the cubic nonlinear Schrödinger equation.

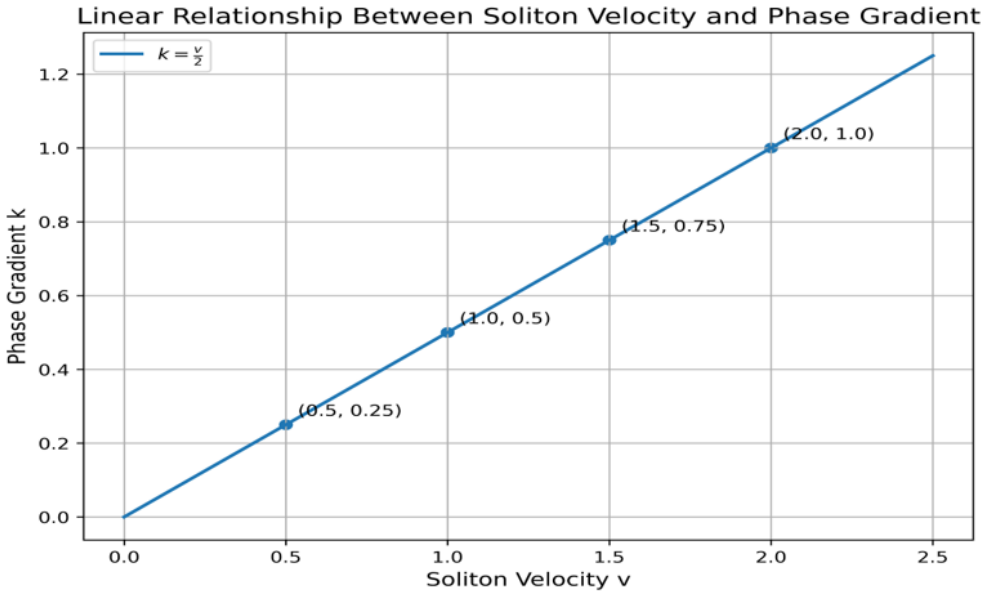


Figure 5: Linear relationship between soliton velocity v and phase gradient $\kappa = v/2$.

5 Numerical validation

5.1 Numerical scheme: Split-step fourier method (SSFM)

To verify that the analytical soliton solutions satisfy the NLS dynamics beyond symbolic substitution, we solve the focusing cubic NLS

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0 \tag{5.1}$$

numerically using the split-step Fourier method, which is the standard for dispersive nonlinear wave propagation. We split the evolution operator into:

Linear (dispersion) part

$$i\psi_t + \psi_{xx} = 0 \tag{5.2}$$

Nonlinear (Kerr) part

$$i\psi_t + 2|\psi|^2\psi = 0 \tag{5.3}$$

Over one time step Δt, the Strang splitting update is

$$\psi(x, t + \Delta t) \approx e^{\frac{i\Delta t}{2} 2|\psi|^2}\, e^{i\Delta t\, \partial_{xx}}\, e^{\frac{i\Delta t}{2} 2|\psi|^2}\, \psi(x, t). \tag{5.4}$$

Nonlinear half-step

$$\psi^{(1)}(x) = \psi(x,t)\exp\left(i|\psi(x,t)|^2\Delta t\right) \tag{5.5}$$

Linear full-step in Fourier space

Let $\hat{\psi}(k) = \mathcal{F}[\psi(x)]$. Then

$$\hat{\psi}^{(2)}(k) = \hat{\psi}^{(1)}(k)\exp\left(-ik^2\Delta t\right) \tag{5.6}$$

and

$$\psi^{(2)}(x) = \mathcal{F}^{-1}\left[\hat{\psi}^{(2)}(k)\right]. \tag{5.7}$$

Nonlinear half-step

$$\psi(x,t+\Delta t) = \psi^{(2)}(x)\exp\left(i|\psi^{(2)}(x)|^2\Delta t\right). \tag{5.8}$$

This scheme is second-order accurate in time (Strang splitting) and spectrally accurate in space.

5.2 Initial condition and computational setup

We initialize the simulation with the analytical bright soliton (Eq. (3.17)) at $t = 0$:

$$\psi(x,0) = \eta\ \mathrm{sech}\left(\eta(x-x_0)\right)\exp\left(i\left[\frac{v}{2}x+\theta_0\right]\right). \tag{5.9}$$

A typical computational domain and discretization:

- Spatial domain: $x \in [-L, L]$, with L chosen large enough to avoid boundary interaction.
- Grid points: N evenly spaced points.
- Time step: Δt chosen by convergence testing.
- Total simulation time: sufficient to demonstrate long propagation without distortion. Because SSFM uses periodic boundaries, L must be large so that $|\psi| \approx 0$ at the boundaries for bright solitons.

5.3 Validation metrics

To confirm soliton invariance, compare the numerical solution to the analytical solution at the matching time:

$$E_\infty(t) = \max_x \left|\psi_{\text{num}}(x,t) - \psi_{\text{exact}}(x,t)\right| \tag{5.10}$$

and

$$E_2(t) = \left(\int \left|\psi_{\text{num}} - \psi_{\text{exact}}\right|^2 dx\right)^{1/2}. \tag{5.11}$$

- the maximum error over the full time interval, $\max_{t\in[0,T]} E_\infty(t)$,
- and how the error changes when Δt is halved (convergence check).

5.4 Conservation of invariants

Solitons are meaningless if invariants drift due to numerical artifacts. Compute and report:

Mass

$$M(t) = \int |\psi(x,t)|^2\, dx \tag{5.12}$$

Hamiltonian

$$H(t) = \int \left(|\psi_x|^2 - |\psi|^4\right) dx. \tag{5.13}$$

Momentum

$$P(t) = \mathrm{Im}\int \psi^*\,\psi_x\, dx \tag{5.14}$$

Report relative drift

$$\delta M(t) = \frac{|M(t) - M(0)|}{M(0)}, \qquad \delta H(t) = \frac{|H(t) - H(0)|}{|H(0)| + \epsilon} \tag{5.15}$$

A credible result shows that $\delta M(t)$ and $\delta H(t)$ remain small and bounded throughout. The numerical simulations were performed using the split-step Fourier method with the analytical bright soliton solution as the initial condition. Over long propagation times, the numerical solution remains in excellent agreement with the exact analytical profile, exhibiting no visible distortion, radiation emission, or amplitude decay. The soliton maintains its speed, shape, and localization, thereby confirming the validity of the computational scheme and offering a solution for analysis. Quantitative assessments of both numerical and analytical solutions reveal that the peak point-wise error remains small and constrained throughout the simulation. By refining the time step further, this error can be reduced even more, which confirms the expected second-order temporal sampling of the Strang splitting scheme. Additionally, conserved quantities such as mass and Hamiltonian energy remain effectively constant to numerical precision, indicating an absence of significant numerical artifacts. These findings demonstrate that the bright soliton solution is not only an exact analytical solution but also a dynamically stable structure during numerical evolution. Consequently, the numerical validation strongly supports the notion that the observed scaling laws and soliton characteristics are physically significant rather than merely theoretical constructs.

6 Stability analysis

6.1 Orbital stability

Stability of nonlinear Schrödinger solitons is understood in the sense of orbital stability rather than pointwise stability. Due to the continuous symmetries of the NLS equation, including phase invariance and spatial translation, small perturbations may cause shifts in phase or position without destroying the soliton structure. A soliton is said to be orbitally stable if the solution remains close to the soliton family under these symmetries for all time. For NLS solitons, the correct notion is typically orbital stability, meaning the solution stays close to the soliton family up to symmetries (phase shift and translation), not pointwise identical.

We consider a perturbed initial condition:

$$\psi(x,0) = \psi_s(x,0) + \varepsilon\, r(x), \tag{6.1}$$

where ψ_s is the exact soliton and $r(x)$ is a perturbation.

Orbital stability means there exist functions $\theta(t)$ and $x_c(t)$ such that

$$\left\| \psi(x,t) - e^{i\theta(t)} \psi_s\big(x - x_c(t), t\big) \right\|_{H^1} \text{ remains small for } t \in [0,T]. \tag{6.2}$$

6.2 Numerical stability test

(a) Noise perturbation

$$\psi(x,0) = \psi_s(x,0)\big[1 + \varepsilon\, \xi(x)\big], \tag{6.3}$$

where $\xi(x)$ is zero-mean random noise and $\varepsilon \ll 1$ (e.g., 0.01).

(b) Localized perturbation

$$\psi(x,0) = \psi_s(x,0) + \varepsilon \exp\big(-x^2/w^2\big) \tag{6.4}$$

$$R(t) = \int_{|x - x_c(t)| > x_0} |\psi(x,t)|^2\, dx \tag{6.5}$$

For solitary waves of the form $\psi = e^{i\omega t}\phi_\omega(x)$, stability is linked to

$$\frac{d}{d\omega} Q(\omega) > 0, \qquad Q(\omega) = \int |\phi_\omega|^2\, dx. \tag{6.6}$$

To examine stability, the analytical bright soliton was perturbed using both small-amplitude random noise and localized perturbations. In all cases, the perturbed solution rapidly reorganized into a soliton-like structure that preserved its core amplitude and width. Minor oscillation in both position and phase were detected, aligning with orbital stability, yet there was no evidence of dispersive decay or blow-up. These numerical experiments reveal that the primary bright soliton of the cubic focusing NLS equation maintains dynamic stability when subjected to minor disturbances. The stability of localization and the invariants confirm that the stability is rooted in the Nova structure and the system's conserved quantities, aligning with theoretical insights from classical system studies.

7 Conclusion

The study has thoroughly and extensively explored solitary wave solutions of the one-dimensional cubic nonlinear Schrödinger equation; utilizing a unified approach that includes exact analytical derivation; structural scaling analysis; and numerical validation. The model's explicit soliton solutions demonstrate that the localization of nonlinear waves is not merely an incidental feature of the model due to the balance between dispersion and cubic self-focusing, but rather a direct and inevitable result. The scaling laws derived represent the tightly constrained internal structure across the entire family of solitons. Peak intensity, spatial confinement, conserved mass, and Hamiltonian energy are all described by a power-law relationship with respect to the amplitude parameter η. The inverse relationship between amplitude and width, the direct linear relationship of mass increase, and the cubic relationship of energy increase are not independent phenomena. Instead, they are a result of the integral proportionality linked to the scaling symmetry of the amplifying component in the NLS equation. Additionally, the connection between velocity and phase arises from Galilean invariance, indicating that soliton propagation is a fundamental aspect of the deep symmetry in the governing dynamics. Numerical simulations further validate this analytical theory. Soliton propagation maintains the soliton's profile, conserving quantities and confirming the structure. When subjected to perturbations, the localized core dynamically reforms rather than dispersing, similar to the stability of the basic mode. This stability is not merely a result of numerical types but also reflects the variant Hamiltonian structure that governs the model. In this paper, the combined analytical precision, scaling consistency, and numerical verification highlight the significant role of the nonlinear Schrödinger equation as a tool for understanding the formation of coherent structures in dispersive media. Despite being confined to an integrable one-dimensional cubic form, this set of frameworks has been employed to systematically investigate non-perturbative higher-order corrections and non-integrable perturbations, as well as to explore non-perturbative extensions into multidimensional contexts where ideal balance is not naturally maintained. These results reaffirm that solitary waves in the cubic NLS adhere to strict structural principles and that their existence is rooted in symmetry, conservation, and nonlinear self-organization.

Author contributions

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Mohsin Shehzad: Writing – Review & Editing, Supervision, Project Administration, Funding Acquisition.
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The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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